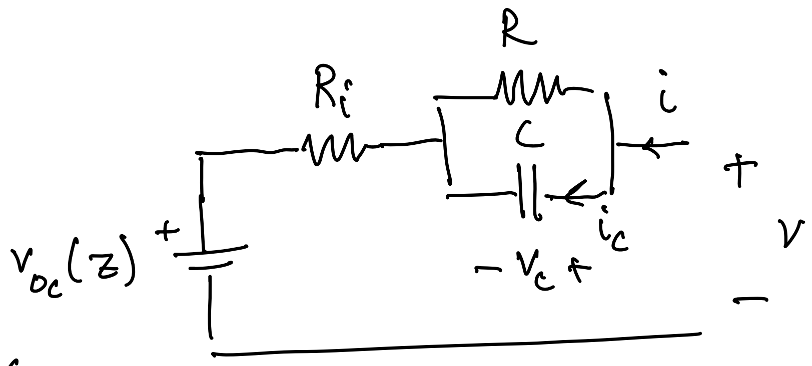
Derivation of an equivalent circuit model (ECM) in State-space form



z = SOC

states: z, v_c
input: i
output: v

KVL: $v = v_{oc}(z) + R_i i + v_c$ (1)

Capacitance: $i_c = C \cdot \dfrac{dv_c}{dt}$ (2)

eliminate i_c

KVL: $v_c = R(i - i_c)$ (3)

SOC: $z = \dfrac{1}{Q}\int i\,dt \Rightarrow \dfrac{dz}{dt} = \dfrac{1}{Q} \cdot i$ (4)

Solve i_c in (3): $v_c = Ri - Ri_c \Leftrightarrow Ri_c = Ri - v_c \Leftrightarrow i_c = i - \dfrac{v_c}{R}$

Substitute in (2):

$$\frac{dv_c}{dt} = \frac{1}{C} \cdot i_c = \frac{1}{C}\left(i - \frac{v_c}{R}\right)$$

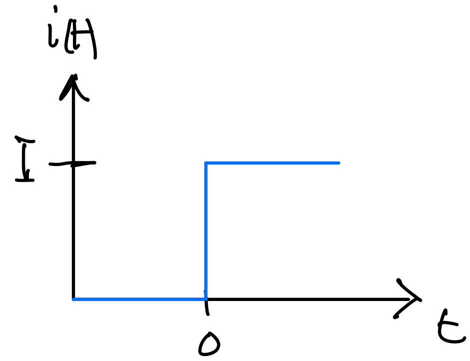
ECM in ss-form

$$\begin{cases} \dfrac{dv_c}{dt} = -\dfrac{v_c}{RC} + \dfrac{i}{C} & (5) \\[2ex] \dfrac{dz}{dt} = \dfrac{1}{Q} \cdot i & (6) \\[2ex] v = v_c + v_{oc}(z) + R_i i & (7) \end{cases}$$

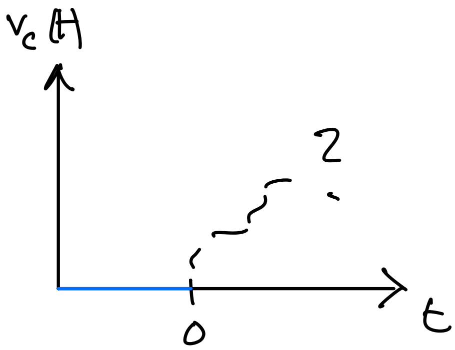
p20 Solving the model (5)(7) for a current step

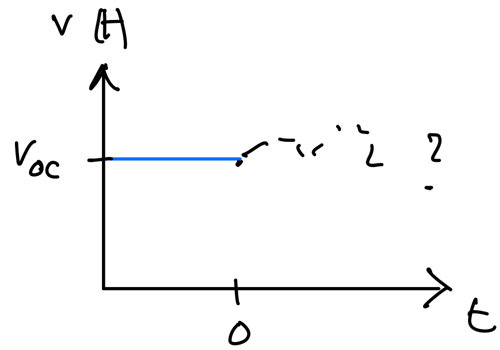
Consider a current step:

$$i(t) = \begin{cases} 0 & t < 0 \\ I & t \geq 0 \end{cases} \qquad v_c(0) = 0$$



p21 Analytical solution for v_c and v:





Equation (5) becomes the inhomogenous differential eq

$$\frac{dv_c}{dt} + \frac{1}{RC} v_c = \frac{I}{C} \qquad (t \geq 0) \qquad (8)$$

The solution is

$$v_c = v_{c,h} + v_{c,p} \qquad (9)$$

where $v_{c,h}$ is the general solution to $\frac{dv_c}{dt} + \frac{1}{RC} v_c = 0$

$$v_{c,h} = ke^{-\frac{1}{RC}t} \qquad \frac{dv_{c,h}}{dt} = -\frac{1}{RC} \cdot \underbrace{ke^{-\frac{1}{RC}t}} = -\frac{1}{RC} v_{c,h}$$

$v_{c,p}$ a particular solution to the inhomogenous equation (8).

Choose $V_{c,p}$ to be a constant such that

$$\frac{1}{R\cancel{C}} \cdot V_{c,p} = \frac{I}{\cancel{C}} \iff V_{c,p} = RI$$

The solution (9) is

$$V_c = k e^{-\frac{1}{RC} \cdot t} + RI$$

k is computed from the initial condition $V_c(0) = 0$

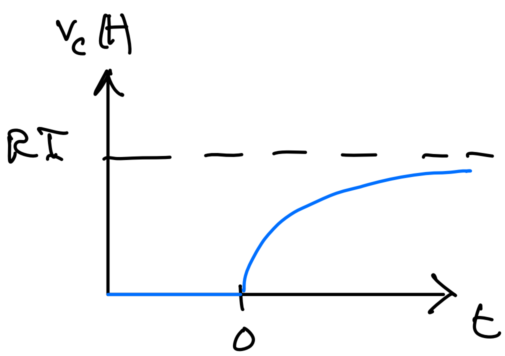
$$0 = k \cdot \underbrace{e^0}_{=1} + RI \iff k = -RI$$

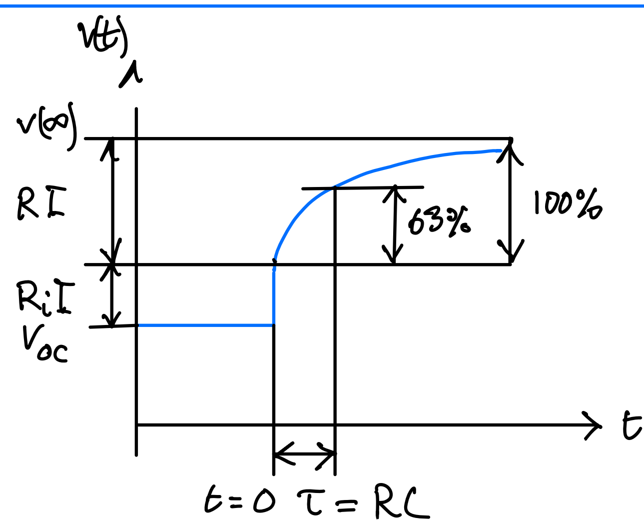
The solution of (8) is

$$V_c = RI\left(1 - e^{-\frac{1}{RC} t}\right)$$

and from (7) we get:

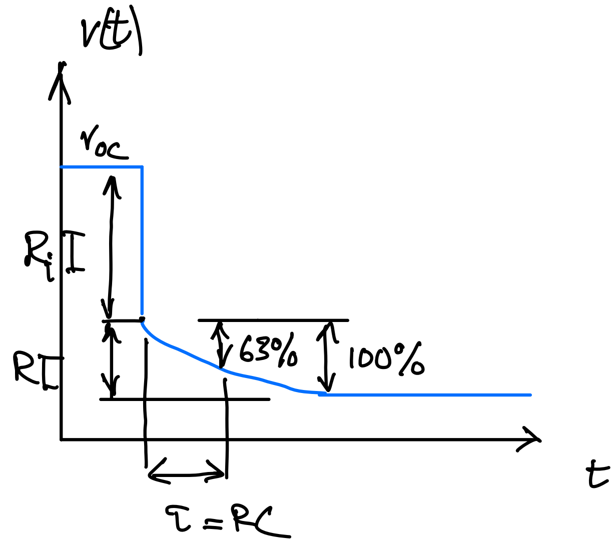
$$\boxed{V = \begin{cases} V_{oc} + RI\left(1 - e^{-\frac{1}{RC} t}\right) + R_i I & t \geq 0 \\ V_{oc} & t < 0 \end{cases} \quad (10)}$$





Computation of model parameters (from step response)

Parameters: V_{oc}, R_i, R, C



V_{oc}: $v(0-) = V_{oc}$

R_i: $v(0) = V_{oc} + R_i I$

$v(0) - V_{oc} = R_i I$

$$R_i = \frac{v(0) - V_{oc}}{I}$$

R: $v(\infty) = V_{oc} + RI + R_i I$

$v(\infty) - (V_{oc} + R_i I) = RI$

$$R = \frac{v(\infty) - (V_{oc} + R_i I)}{I}$$

C: The time constant τ of the system is the time $t=\tau$ it takes to reach $1-e^{-1} = 63\%$ of the final value after a step input is applied.

τ is measured in the figure.

The exponent is -1 at $t=\tau$:

$$-1 = -\frac{1}{RC}\cdot\tau \Leftrightarrow \tau = RC \Leftrightarrow C = \frac{\tau}{R}$$

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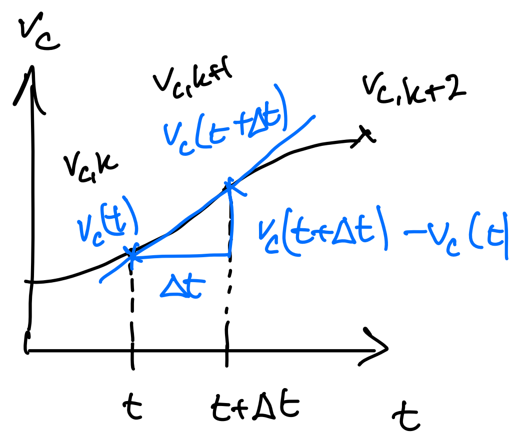
ECM in time discrete form

Time continous form:

$$\begin{cases} \frac{dv_c}{dt}(t) = -\frac{v_c(t)}{RC} + \frac{i(t)}{C} \\ v(t) = v_c(t) + v_{oc} + R_i\, i(t) \end{cases}$$

Consider sampled signals :

$$x_k = x(t)$$
$$x_{k+1} = x(t+\Delta t)$$
$$x_{k+2} = x(t+2\Delta t)$$
$$\vdots$$



Euler forward approximation of the derivative:

$$\frac{dv_c}{dt}(t) \approx \frac{v_c(t+\Delta t) - v_c(t)}{\Delta t} = \frac{v_{c,k+1} - v_{c,k}}{\Delta t}$$

$$\begin{cases} \frac{v_{c,k+1} - v_{c,k}}{\Delta t} = -\frac{v_{c,k}}{RC} + \frac{i_k}{C} \\ v_k = v_{c,k} + v_{oc,k} + R_i\, i_k \end{cases}$$

$$\begin{cases} v_{c,k+1} = v_{c,k}\left(1 - \frac{\Delta t}{RC}\right) + \frac{i_k}{C}\Delta t \\ v_k = v_{c,k} + v_{oc,k} + R_i\, i_k \end{cases}$$

Linear Regression

Put the model

$$\begin{cases} \dfrac{V_{c,k+1} - V_{c,k}}{\Delta t} = -\dfrac{V_{c,k}}{RC} + \dfrac{i_k}{C} & (11) \\ V_k = V_{c,k} + V_{oc,k} + R_i i_k & (12) \end{cases}$$

in the form

$Y = X\theta$ where Y, X are known matrices and θ parameters to be estimated.

Row k is expressed as

$$y_k = x_{k1}\theta_1 + x_{k2}\theta_2 + x_{k3}\theta_3$$

Let $\tilde{v}_k = v_k - v_{oc,k}$ to eliminate $v_{oc,k}$

$$(12) \implies \tilde{v}_k = v_{c,k} + R_i i_k \qquad (12')$$

Eliminate $v_{c,k}$, $v_{c,k+1}$ in (11) and (12'):

Solve $v_{c,k}$ in (12'): $\quad v_{c,k} = \tilde{v}_k - R_i i_k$

Substitute in (11):

$$\frac{\tilde{v}_{k+1} - R_i i_{k+1} - (\tilde{v}_k - R_i i_k)}{\Delta t} = -\frac{1}{RC}(\tilde{v}_k - R_i i_k) + \frac{i_k}{C}$$

The linear regression is

$$\underbrace{\frac{\tilde{v}_{k+1} - \tilde{v}_k}{\Delta t}}_{= y_k} = \underset{\theta_1}{R_i} \underbrace{\frac{i_{k+1} - i_k}{\Delta t}}_{x_{k1}} + \underset{\theta_2}{\frac{1}{RC}} \underset{x_{k2}}{(-\tilde{v}_k)} + \underbrace{\left(\frac{R_i}{RC} + \frac{1}{C}\right)}_{\theta_3} \underset{x_{k3}}{i_k}$$

The matrices are

$$Y = \begin{bmatrix} \frac{\tilde{v}_2 - \tilde{v}_1}{\Delta t} \\ \vdots \\ \frac{\tilde{v}_n - \tilde{v}_{n-1}}{\Delta t} \end{bmatrix} \qquad X = \begin{bmatrix} \frac{i_2 - i_1}{\Delta t} & -\tilde{v}_1 & i_1 \\ \vdots & & \\ \frac{i_n - i_{n+1}}{\Delta t} & -\tilde{v}_{n-1} & i_{n-1} \end{bmatrix}$$

In Matlab : $\theta = X \backslash Y$

Model parameters can be computed by solving :

$$\begin{cases} \theta_1 = R_i \\ \theta_2 = \frac{1}{RC} \\ \theta_3 = \frac{R_i}{RC} + \frac{1}{C} \end{cases}$$

<u>Solution:</u>

<u>R_i</u> : $R_i = \theta_1$

<u>C</u> : $\theta_3 = \theta_1 \cdot \theta_2 + \frac{1}{C} \Leftrightarrow C = (\theta_3 - \theta_1\theta_2)^{-1}$

<u>R</u> : $R = \frac{1}{C\,\theta_2} = \frac{\theta_3 - \theta_1\theta_2}{\theta_2} = \frac{\theta_3}{\theta_2} - \theta_1$