Number theory, Talteori 6hp, Kurskod TATA54, Provkod TEN1
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All problems are worth 3 points. To receive full points, a solution needs to be complete. Prove your assertions, indicate which theorems from the textbook that you have used, and include all auxillary calculations.

No aids, no calculators, tables, nor textbooks.

3:8p 4:13p 18:5p

1) What is the remainder when $16!$ is divided by 19?

2) Calculate $\gcd(5k+3, 3k+2)$ for all positive integers k.

3) Find all solutions in positive integers of the Diophantine equation

$$x^2 + 3y^2 = z^2$$

4) Determine $\left(\frac{-3}{p}\right)$ when p is an odd prime.

5) The Gaussian integer α is not a unit, nor is it a Gaussian prime. Show that there exists a Gaussian integer β with $\beta|\alpha$, $1 < N(\beta) \leq \sqrt{N(\alpha)}$.

6) The rational number $r = \frac{151}{115}$ has a rational approximation $s = a/b$, a, b positive integers, such that

   (a) $1 < b < 10^2$,
   (b) $|r - s| < 10^{-3}$

   Find such an s.

7) Let $\tau(k)$ denote the number of positive divisors of the positive integer k. Show that τ is multiplicative. Calculate, for each positive integer n,

$$\sum_{k=1}^{n} \tau(k) - \sum_{k=1}^{n} \lfloor \frac{n}{k} \rfloor$$

   Here, for each $x \in \mathbf{R}$, $\lfloor x \rfloor$ denotes the largest integer $\leq x$.

   (Hint: let $g(\ell, k) = \begin{cases} 1 & \ell|k \\ 0 & \text{otherwise} \end{cases}$. Then $\lfloor \frac{k}{\ell} \rfloor - \lfloor \frac{k-1}{\ell} \rfloor = g(\ell, k)$. Telescope.)